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TWO- AND THREE-DIMENSIONAL RESULTS FOR ROTATIONALLY SYMMETRIC DEFORMATIONS OF CIRCULAR CYLINDRICAL SHELLS

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Abstract—Previous considerations by asymptotic expansion procedures of the relation between elasticity theory results and thin-shell theory results for the case of rotationally symmetric deformations of an edge-loaded semi-infinite circular cylindrical shell are supplemented by an analysis of this problem for a shell possessing a limiting-type orthotropy, such that transverse normal strains vanish identically. It is shown that assuming this kind of orthotropy has the important benefit of allowing the derivation of exact expressions for the edge zone solution contribution, when such exact expressions are not possible for the problem of the shell with more general properties of the material. One result of the present analysis is an answer to the following question. Given a shell with arbitrarily prescribed edge displacements (compatible with the assumed type of orthotropy), what is the asymptotically exact form of the corresponding conditions for this same problem, treated within the framework of two-dimensional thin-shell theory?

INTRODUCTION

We return once more to the problem of rotationally symmetric deformations of a semi-infinite circular cylindrical shell as the simplest non-trivial example of the relation between three-dimensional elasticity-theory analysis and two-dimensional thin-shell-theory analysis. The first paper on this subject[1] considered the problem of the asymptotic determination of a class of "interior" solutions for the given three-dimensional boundary value problem, and the derivation therefrom of a system of two-dimensional shell-theory equations including the formulation of shell-theory boundary condition statements from given three-dimensional elasticity-theory statements of such conditions, for the case that these conditions were stress boundary conditions. A subsequent paper by Reiss[2] extended this work by considering complete asymptotic solutions, including interior solutions and "edge-zone" solution contributions. The results obtained in this manner confirmed the conclusions in [1] in regard to the problem of two-dimensional shell theory, while at the same time supplying significant additional insights in regard to the nature of the relation between two- and three-dimensional theory, with these insights having meanwhile been extended and generalized in important ways by various other workers, in particular by Goldenweiser[3].

One of the difficulties encountered in the use of an edge-zone solution contribution, as done by Reiss[2], consists in the fact that the relevant two-dimensional boundary value problem for a bi-harmonic function defined in a semi-infinite strip cannot, for some important cases including the case of pure traction conditions and of pure displacement conditions, be solved in closed form, and to the extent that this is the case the asymptotic results which are obtained remain approximate rather than exact.

Given the impossibility of a closed-form solution of the relevant bi-harmonic problem, as well as the apparent absence of results for the case of pure displacement edge condition cases, we have recently considered the complete problem by combining interior asymptotic expansions, Rayleigh–Ritz type edge-zone solution contributions, and upper and lower bound formulas through use of the principles of minimum potential and complementary energies[4]. The principal results of this analysis consisted in the derivation of upper and lower bounds for the values of influence coefficients involved in the solution of the semi-infinite circular cylindrical shell problem with prescribed edge tractions or prescribed edge displacements. All of these bound results were such as to imply the determination of *exact* results for the solution of the three-dimensional problem by means of two-dimensional theory, insofar as the *leading* terms in an expansion of the solution of the three-dimensional problem in powers of wall thickness h to shell radius ratio a were concerned. We also determined *supplementary* terms for such an expansion, of relative order h/a (including terms of order $(h/a)^{1/2}$ which are encountered for some classes of edge conditions), with these supplementary bound terms being such that in some cases there was coincidence between upper and lower bound results so that, in effect, a determination of *exact* results, up to terms of relative order h/a, was accomplished.

In order to understand the meaning of these results, as well as the significance of the analysis which follows it is convenient to interpret the leading-term results as exact results for an infinitely-thin shell (i.e. for a shell for which $h/a \to 0$), with the supplementary terms representing the effects of finite thickness. There are altogether three distinct effects of finite thickness. The first of these is a geometrical effect, having to do with the change of width of shell elements with distance from the middle surface (so that this effect is absent for the special case of a flat plate). It is known that this effect is taken account of properly (assuming absence of the other two) in a refined two-dimensional shell theory associated with the names of Flügge, Lurie and Byrne The second effect is the effect of transverse normal stress deformability (which is absent for the case of a limiting-type orthotropic material unable to sustain transverse normal strains). It has earlier been shown that this effect is of the same order of magnitude as the geometrical FLB effect[1], and our recent work[4] indicates that for some classes of edge conditions this effect comes out to be of relative order $(h/a)^{1/2}$ (without these terms being of numerical significance, however, in comparison with the co-existing h/a-order effects).

The third of the effects of finite thickness is the effect of transverse shear deformability. This effect too comes out to be of relative order h/a. We are not concerned here with the consequences of this effect, in regard to the order of the differential-equation system and to the number of the associated boundary conditions, as discussed most simply in recent work dealing with the subject of plates[5]. Rather, we are concerned with this effect from the point of view of its relative numerical dominance in comparison with the other two, as revealed by our upper and lower bound calculations[4].

Having previously obtained upper and lower bound results for the three effects of geometry, transverse normal stress deformability, and transverse shear deformability, with these three effects being additive up to orders of magnitude which are of primary interest, we now undertake an asymptotic analysis of two of the three effects, these being the effects of transverse shear deformability and of geometry. Our analysis is based on recognition of the fact that it is possible to derive *exact* solutions for the two-dimensional semi-infinite strip problem governing the edge-zone solution contribution, upon assuming a limiting-type orthotropy in such a manner that transverse normal strains vanish identically.

Having the existence of these exact solutions for the edge-zone contributions involved in the asymptotic expansion procedure, we are now in a position to verify and, in principle, to refine the results of our upper and lower bound analysis. Beyond this, we are able to obtain results for types of boundary conditions which do not fall within the scope of the indicated bound solutions. To mention a specific example, our analysis permits us to solve a problem which has long been of interest to us but for which until now no rational solution has come to our attention. The problem is as follows. Given a semi-infinite circular cylindrical shell, with *arbitrarily* prescribed edge *displacements* as loading conditions. To be determined is the *asymptotically exact* form of the corresponding conditions of the first-order interior solution contribution, to wit, the appropriate form of the corresponding boundary conditions for this same problem, treated within the frame work of standard two-dimensional thin shell theory.

FORMULATION OF THE PROBLEM

We take as differential equations for symmetrical deformations of circular cylindrical bodies a system consisting of the equilibrium equations

$$r\sigma_{x,x} + (r\tau)_{,r} = 0, \quad r\tau_{,x} + (r\sigma_r)_{,r} - \sigma_\theta = 0, \tag{2.1}$$

in conjunction with stress–strain (displacement) relations of the form

$$u_{,x} = \frac{\sigma_x - \nu\sigma_\theta}{E} - \nu_r \frac{\sigma_r}{E_m}, \quad \frac{v}{r} = \frac{\sigma_\theta - \nu\sigma_x}{E} - \nu_r \frac{\sigma_r}{E_m},$$

$$v_{,r} = \frac{\sigma_r}{E_r} - \nu_r \frac{\sigma_x + \sigma_\theta}{E_m}, \quad u_{,r} + v_{,x} = \frac{\tau}{G}, \tag{2.2}$$

where $E_m = (EE_r)^{1/2}$, with positive E, E_r and G, and with the additional strain energy positive-definiteness conditions $\nu^2 < 1$ and $2\nu_r^2 < 1 - \nu$.

The system (2.1) and (2.2) is to be solved in the region $0 \le x < \infty$, $a - c \le r \le a + c$ subject to face boundary conditions

$$r = a \pm c: \qquad \sigma_r = 0, \qquad \tau = 0, \tag{2.3}$$

subject to edge boundary conditions "at infinity" which for all cases are taken in the form

$$x \to \infty: \qquad u = 0, \qquad v = 0 \tag{2.4}$$

and subject to edge boundary conditions at the loaded edge of the shell, of the form

$$x = 0: \quad \begin{cases} u_{,r} = \bar{u}_{,r}, & or \quad \sigma_x = \bar{\sigma}_x, \\ v = \bar{v}, & or \quad \tau = \bar{\tau}, \end{cases} \tag{2.5}$$

with the r.h.s. in these four relations being *prescribed* functions of r, subject only to the restriction that $\int_{a-c}^{a+c}(r/a)\bar{\sigma}_x \, dr = 0$.

Within the frame work of the above class of three-dimensional problems (which because of the assumed absence of any θ-dependence of the solutions formally reduces to a class of two-dimensional problems) we are particularly concerned in asymptotic reductions to two-dimensionality (with this reduction here formally to one-dimensionality) for the determination of the weighted stress averages

$$M_x = \int_{a-c}^{a+c} (r/a)\sigma_r(r-a)\, dr, \quad Q = \int_{a-c}^{a+c} (r/a)\tau \, dr, \tag{2.6}$$

and for the determination of displacement measures such as

$$V = v(x, 0), \qquad \beta = u_{,r}(x, 0), \tag{2.7}$$

with these reductions being of technical significance for "sufficiently small" values of the wall thickness–diameter ratio c/a of the shell.

In association with the derivation of a system of two-(here one-) dimensional differential equations for the quantities M_x, Q, V, β it is necessary to also derive a system of suitable edge conditions, involving the functions $\bar{u}_{,r}$, $\bar{v}$, $\bar{\sigma}_x$, $\bar{\tau}$ which appear in eqns (2.5). One expects, and this has been shown to be true in [1] and [2] that insofar as the traction condition portions in (2.5) are concerned the equivalent lower-dimensional edge conditions are of the form

$$x = 0: \qquad M_x = \bar{M}_x, \qquad Q = \bar{Q}, \tag{2.8a}$$

with $\bar{M}_x$ and $\bar{Q}$ defined in terms of $\bar{\sigma}_x$ and $\bar{\tau}$ in accordance with eqns (2.6).

One also expects, and this is generally considered to be correct, that when the displacement condition portions in eqns (2.5) are such that $\bar{u}_{,r}$ *as well as* $\bar{v}$ *are independent of* r, then the equivalent lower-dimensional edge conditions are of the form

$$x = 0: \qquad V = \bar{v}, \qquad \beta = \bar{u}_{,r}. \tag{2.8b}$$

We will, in what follows, re-confirm the asymptotic validity of the above expectation, within the frame work of the restrictions associated with the nature of the analysis which is here carried out. Over and above this result, however, we will establish the form of the edge conditions for "effective" measures V and β, not necessarily identical with these measures as defined in eqns (2.7), which are valid in the event that $\bar{u}_{,r}$ and $\bar{v}$ are other than independent of r. An example of this nature which will be considered explicitly is the case where $\bar{u}$ is assumed to be proportional to $(r-a)^3$ in place of the usually assumed linear distribution.

DIFFERENTIAL EQUATIONS AND BOUNDARY CONDITIONS FOR TRANSVERSELY INEXTENSIONAL DEFORMATIONS

The case of transversely inextensional deformations is given upon setting

$$E_r = \infty \tag{3.1}$$

in the stress–strain relations (2.2). These may therewith be written in the form

$$Eu_{,x} = \sigma_x - \nu\sigma_\theta, \qquad Er^{-1}v = \sigma_\theta - \nu\sigma_x,$$
$$v_{,r} = 0, \qquad G(u_{,r} + v_{,x}) = \tau, \tag{3.2}$$

where it will be assumed from here on that E, ν and G are independent of x and r.

Our first conclusion is now that the transverse displacement component does not vary across the thickness,

$$v = V(x) \tag{3.3}$$

and that, associated therewith, the transverse normal stress component σ_r assumes the character of a reactive quantity.

In order to solve the remaining boundary value problem, we begin by satisfying the first of the equilibrium equations (2.1) by means of a stress function Ψ, in the form

$$r\sigma_x = a\Psi_{,r}, \qquad r\tau = -a\Psi_{,x}, \tag{3.4}$$

where the factor a on the r.h.s. has been introduced to make the writing of some of the developments which follow somewhat more convenient.

We next use the second of the stress–strain relations (3.2), in conjunction with eqn (3.3) in order to write for the circumferential normal stress,

$$r\sigma_\theta = \nu a\Psi_{,r} + EV. \tag{3.5}$$

With the above expressions for σ_θ and τ, we obtain from the second equilibrium equation in (2.1) as an expression for the transverse normal stress σ_r which satisfies the condition of vanishing σ_r for $r = a - c$

$$r\sigma_r = a\int_{a-c}^{r} (\Psi_{,xx} + \nu r^{-1}\Psi_{,r})\,\mathrm{d}r + EV\int_{a-c}^{r} r^{-1}\,\mathrm{d}r. \tag{3.6}$$

The condition that σ_r must also vanish for $r = a + c$ gives as one of two equations connecting the two functions $\Psi(x, r)$ and $V(x)$

$$\int_{a-c}^{a+c} \left(\Psi_{,xx} + \frac{\nu}{r}\Psi_{,r}\right)\mathrm{d}r + \frac{EV}{a}\ln\frac{a+c}{a-c} = 0. \tag{3.7}$$

A second equation connecting Ψ and V follows upon expressing $u_{,r}$ in terms of Ψ and V in accordance with the last relation in (3.2), in conjunction with (3.3) and (3.4), as

$$u_{,r} = -V_{,x} - \frac{a}{r}\frac{\Psi_{,x}}{G}, \tag{3.8}$$

and by combining this result with the first relation in (3.2), written in the form $Eu_{,xr} = \sigma_{x,r} - \nu\sigma_{\theta,r}$, with σ_x and σ_θ taken from eqns (3.4) and (3.5). The ensuing differential equation may be written in the form

$$\frac{1-\nu^2}{E}\left(\Psi_{,rr} - \frac{\Psi_{,r}}{r}\right) + \frac{1}{G}\Psi_{,xx} + \frac{r}{a}V_{,xx} + \frac{\nu}{ra}V = 0. \tag{3.9}$$

Having eqns (3.7) and (3.9) it remains to state boundary conditions in terms of Ψ and V, in accordance with the remaining face boundary conditions in (2.3), which concern τ, and in accordance with the edge conditions as stated in eqns (2.4) and (2.5).

We begin by satisfying the edge condition at infinity by stipulating

$$\Psi(\infty, r) = 0, \qquad V(\infty) = 0. \tag{3.10}$$

With this, and with observation of the relation $\int_{a-c}^{a+c}(r/a)\sigma_x\,dr = 0$, the face boundary conditions $\tau(x, a \pm c) = 0$ take on the form

$$\Psi(x, a \pm c) = 0 \tag{3.11}$$

Finally, the conditions (2.5) for the loaded edge of the shell become

$$x = 0: \begin{cases} V_{,x} + \dfrac{a}{G}\dfrac{\Psi_{,x}}{r} = -\bar{u}_{,r} \quad \text{or} \quad \dfrac{\Psi_{,r}}{r} = \dfrac{\bar{\sigma}_x}{a}, \\ V = \bar{V} \quad \text{or} \quad \displaystyle\int_{a-c}^{a+c} \Psi_{,r}\,dr = -\bar{Q}. \end{cases} \tag{3.12}$$

We note that in writing the second set of conditions in (3.12), we have taken account of the fact that the assumed properties of the material require that the prescribed transverse displacement $\bar{v}$ be independent of r and so may be written, in consistent fashion, as $\bar{V}$, and that at the same time this specialization of properties implies a sensitivity of the medium to the resultant $\bar{Q}$ of the edge stresses $\bar{\tau}$ only, rather than to the details of a prescribed $\bar{\tau}$-distribution.

STIFFNESS AND FLEXIBILITY COEFFICIENTS

In accordance with our earlier work in [4], we define general *stiffness* coefficients K, with reference to the problem of prescribed edge *displacements*, for the case that $\bar{u}_{,r} = -\beta_0$ and $\bar{V} = -V_0$, by means of relations of the form

$$M_x(0) = K_{\beta\beta}\beta_o - K_{\beta V}V_o, \quad Q(0) = K_{VV}V_o - K_{V\beta}\beta_o, \tag{4.1}$$

where $K_{V\beta} = K_{\beta V}$.

At the same time we define a specialized system of such coefficients for mixed edge condition problems, as follows

$$M_x(0) = K_\beta\beta_0 \quad (\text{when } \bar{\tau} = 0),$$
$$Q(0) = K_V V_o \quad (\text{when } \bar{\sigma}_x = 0). \tag{4.2}$$

Insofar as the problem of prescribed edge *tractions* is concerned, we have earlier [4] defined *flexibility* coefficients C with reference to traction distributions $(r/a)\bar{\sigma}_x = (3M_0/2c^3)(r-a)$ and $(r/a)\bar{\tau} = (3Q_0/4c)[1-(r-a)^2/c^2]$, and weighted edge displacement averages

$$\beta_0^* = -\frac{3}{4c}\int_{a-c}^{a+c}\left[1-\frac{(r-a)^2}{c^2}\right]u_{,r}\,dr,$$
$$V_0^* = -\frac{3}{4c}\int_{a-c}^{a+c}\left[1-\frac{(r-a)^2}{c^2}\right]v\,dr, \tag{4.3}$$

in the form

$$\beta_0^* = C_{MM}M_o + C_{MQ}Q_o, \quad V_o^* = C_{QM}M_o + C_{QQ}Q_o, \tag{4.4}$$

with these relations being "nearly" the inverses of eqns (4.1), upon the identifications $\beta_o^* = \beta_o$, $V_0^* = V_o$, $M_x(0) = M_0$ and $Q(0) = Q_o$.

The principal aim in [4] was the deduction of upper and lower bounds for the coefficients K and C, with these bounds depending on the geometrical parameter c/a, as well as on the materials property parameters E, E/G, E/E_r, ν and ν_r. Among the characteristics of these bound relations the following are noteworthy.

(1) In the limit $c/a \to 0$, our upper and lower bound values coincide with each other and with the corresponding values of what may be called first-approximation classical thin shell theory results.

(2) For sufficiently small finite values of c/a our bound values involve additive terms of relative order $(c/a)^{1/2}$ and of relative order c/a. The terms of order $(c/a)^{1/2}$ represent transverse normal strain effects alone, while the terms of order c/a represent transverse normal strain effects, as well as transverse shearing strain effects, and also geometrical effects of the Flügge–Lurie–Byrne type. The numerically dominant of these three effects is the transverse shearing strain effect, even though this effect does not contain any $(c/a)^{1/2}$-contributions.

Insofar as the determination of stiffness and flexibility coefficients is concerned the analysis enables us to obtain *exact* values of all contributions of relative orders $(c/a)^{1/2}$ and c/a (as well as exact values of contributions of higher order in c/a should we so desire) subject only to the restriction $E_r = \infty$. While it is possible to do this for the entire set of ten coefficients defined in eqns (4.1)–(4.4), the necessary analysis will be carried out in what follows only to the extent of obtaining expressions for $K_{\beta\beta}$, $K_{V\beta}$ and K_β.

THE NON-DIMENSIONALIZED BOUNDARY VALUE PROBLEM

We introduce a non-dimensional axial coordinate ξ and a nondimensional radial coordinate η, measured from the middle surface of the shell, by writing

$$x = b\xi, \qquad r = a + c\eta, \tag{5.1}$$

with b being in the nature of a characteristic length, to be chosen presently.

We further set

$$(1-\nu^2)\Psi = E\Psi_o g(\xi,\eta), \quad V = V_o F(\xi), \tag{5.2}$$

where Ψ_o and V_o remain to be chosen, and we define dimensionless parameters ρ and α in the form

$$\rho = c/a, \quad (1-\nu^2)\alpha^2 = E/G. \tag{5.3}$$

Indicating now differentiations with respect to ξ and η by primes and dots, respectively, the two differential equations (3.7) and (3.9) for Ψ and V take on the following form

$$\Psi_o \int_{-1}^{1} \left[\frac{c^2}{b^2} g'' + \frac{\nu\rho^2}{(1+\rho\eta)^2} g\right] \mathrm{d}\eta + V_o \left[(1-\nu^2)\,\rho \ln\frac{1+\rho}{1-\rho} F\right] = 0, \tag{5.4}$$

$$\Psi_o \left[g^{\cdot\cdot} - \frac{\rho}{1+\rho\eta} g^{\cdot} + \alpha^2 \frac{c^2}{b^2} g''\right] + V_o \left[\frac{c^2}{b^2}(1+\rho\eta)F'' + \frac{\nu\rho^2}{1+\rho\eta} F\right] = 0. \tag{5.5}$$

At the same time the face boundary conditions (3.11) become

$$g(\xi, \pm 1) = 0, \tag{5.6}$$

and the edge boundary conditions at infinity become

$$g(\infty, \eta) = 0, \quad F(\infty) = 0. \tag{5.7}$$

Insofar as the boundary conditions (3.12) at the loaded edge are concerned, we will limit ourselves in what follows to the consideration of just two of a total of eight possible cases, both of them concerning problems with prescribed axial *displacements* as expressed in terms of the slope function $\bar{u}_{,r}$, with the other condition being the displacement condition of vanishing $\bar{V}$, or the traction condition of vanishing $\bar{Q}$. Expressed in terms of g and F this set of conditions takes on the form

$$\frac{\Psi_o}{b}\frac{\alpha^2}{1+\rho\eta} g'(0,\eta) + \frac{V_o}{b} F'(0) = -\bar{u}_{,r} \tag{5.8}$$

$$F(0) = 0 \quad \text{or} \quad \int_{-1}^{1} g'(0,\eta)\,\mathrm{d}\eta = 0. \tag{5.9}$$

Expressions for the stresses σ_x, τ, σ_θ, and for the stress resultant Q and the stress couple M_x follow from eqns (3.4), (3.5) and (2.6) as

$$\sigma_x = \frac{E}{1-\nu^2}\frac{\Psi_o}{c}\frac{g^{\cdot}}{1+\rho\eta}, \quad \tau = -\frac{E}{1-\nu^2}\frac{\Psi_o}{b}\frac{g'}{1+\rho\eta}, \tag{5.10}$$

$$\sigma_\theta = E\frac{V_o}{a}\frac{F}{1+\rho\eta} + \frac{E}{1-\nu^2}\frac{\Psi_o}{c}\frac{\nu g^{\cdot}}{1+\rho\eta}, \tag{5.11}$$

$$M_x = -\frac{E\Psi_o c}{1-\nu^2}\int_{-1}^{1} g\,\mathrm{d}\eta, \qquad Q = -\frac{E}{1-\nu^2}\frac{\Psi_o c}{b}\int_{-1}^{1} g'\mathrm{d}\eta. \tag{5.12}$$

We list additionally as expression for the circumferential stress resultant

$$N_\theta = \int_{a-c}^{a+c} \sigma_\theta\,\mathrm{d}r = EV_o\rho\ln\frac{1+\rho}{1-\rho}F - \frac{E\Psi_o\rho}{1-\nu^2}\int_{-1}^{1}\frac{\nu g\,\mathrm{d}\eta}{(1+\rho\eta)^2}, \tag{5.13}$$

and we shall not concern ourselves, in what follows, with the complementary expressions for σ_r and M_θ.

INTERIOR AND EDGE ZONE SOLUTION CONTRIBUTIONS

We accept as known the fact that the solution to be obtained will, for sufficiently small values of ρ, be composed of two contributions, one of them being an "interior" contribution g_i, F_i, with characteristic length $b = b_i = (ac)^{1/2}$ and the other an "edge zone" contribution g_e, F_e, with characteristic length $b = b_e = c$. We note that in this way $F_i = F_i(\xi_i)$ and $F_e = F_e(\xi_e)$, etc. but we shall refrain from making the distinction between ξ_i and ξ_e explicit, as this will cause no difficulty as the analysis proceeds. With ξ_i, ξ_e and η defined in this manner, and in view of the form of the differential equations and boundary conditions for g and F, we are in a position to stipulate the basic order of magnitude relations

$$g_i, g_e, F_i, F_e, g_i^{\cdot}, g_e^{\cdot}, g_i', g_e', F_i', F_e' = O(1), \tag{6.1}$$

and we shall write, in place of eqns (5.2), so as to make the distinction between the two solution contributions explicit

$$(1-\nu^2)\Psi = E(\Psi_i g_i + \Psi_e g_e), \quad V = V_i F_i + V_e F_e, \tag{6.2}$$

with the form of the two differential equations for the set g_i, F_i and for the set g_e, F_e being distinct, to the extent that this is required to account for the difference between b_i and b_e.

Setting $b = b_i = (ac)^{1/2}$, we obtain from eqns (5.4) and (5.5)

$$\Psi_i\rho\int_{-1}^{1}\left[g_i'' + \frac{\nu\rho}{(1+\rho\eta)^2}g_i\right]\mathrm{d}\eta + V_i\rho^2\left[\frac{1-\nu^2}{\rho}\ln\frac{1+\rho}{1-\rho}F_i\right] = 0, \tag{6.3}$$

$$\Psi_i\left[g_i^{\cdot\cdot} - \frac{\rho}{1+\rho\eta}g_i^{\cdot} + \alpha^2\rho g_i''\right] + V_i\rho\left[(1+\rho\eta)F_i'' + \frac{\nu\rho}{1+\rho\eta}F_i\right] = 0. \tag{6.4}$$

Setting $b = b_e = c$, we obtain instead

$$\Psi_e\int_{-1}^{1}\left[g_e'' + \frac{\nu\rho^2}{(1+\rho\eta)^2}g_e\right]\mathrm{d}\eta + V_e\rho^2\left[\frac{1-\nu^2}{\rho}\ln\frac{1+\rho}{1-\rho}F_e\right] = 0, \tag{6.5}$$

$$\Psi_e\left[g_e^{\cdot\cdot} - \frac{\rho}{1+\rho\eta}g_e^{\cdot} + \alpha^2 g_e''\right] + V_e\left[(1+\rho\eta)F_e'' + \frac{\nu\rho^2}{1+\rho\eta}F_e\right] = 0. \tag{6.6}$$

In view of the difference between ξ_i and ξ_e, the functions g_i and g_e must individually satisfy the face boundary conditions

$$\eta = \pm 1: \quad g_i = 0, \quad g_e = 0, \tag{6.7}$$

and for the same reason the edge conditions at infinity (5.7) must be satisfied individually,

$$\xi = \infty: \quad g_i = 0, \quad g_e = 0, \quad F_i = 0, \quad F_e = 0. \tag{6.8}$$

With g_i and F_i, and g_e and F_e, sofar being subject to a system of uncoupled requirements, there remains only the system edge loading conditions (5.8) and (5.9) to accomplish the necessary coupling for the determination of the two sets of functions. We find that eqn (5.8), again with $b_i = (ac)^{1/2}$ and $b_e = c$, takes on the form

$$\frac{\alpha^2}{1+\rho\eta}[\Psi_i\rho^{1/2}g_i'(0,\eta) + \Psi_e g_e'(0,\eta)] + V_i\rho^{1/2}F_i'(0) + V_e F_e'(0) = -c\bar{u}_{,r}, \tag{6.9}$$

and eqns (5.9) take on the form

$$V_iF_i(0) + V_eF_e(0) = 0, \tag{6.10a}$$

or

$$\int_{-1}^{1}[\Psi_i\rho^{1/2}g_i'(0,\eta) + \Psi_e g_e'(0,\eta)]\,d\eta = 0. \tag{6.10b}$$

Having the relations (6.2)–(6.10) our next step is an appropriate disposition of the four scale factors Ψ_i, Ψ_e, V_i, V_e. We begin by observing that in order to be able to satisfy the face boundary conditions (6.7) in such a way that not only the functions g_i and g_e, but also the functions F_i and F_e are involved it will be necessary to have the terms with g'' in eqns (6.4) and (6.6) of the same order of magnitude as the terms with F'' in these equations. Accordingly, we set

$$\Psi_i = \rho V_i, \quad \Psi_e = V_e. \tag{6.11}$$

Introduction of (6.11) into the edge conditions (6.9) and (6.10) then gives further

$$V_i\rho^{1/2}\left[F_i'(0) + \frac{\alpha^2\rho}{1+\rho\eta}g_i'(0,\eta)\right] + V_e\left[F_e'(0) + \frac{\alpha^2}{1+\rho\eta}g_e'(0,\eta)\right] = -c\bar{u}_{,r}, \tag{6.12}$$

with (6.10a) remaining unchanged, and eqn (6.10b) assuming the form

$$\int_{-1}^{1}[V_i\rho^{3/2}g_i'(0,\eta) + V_e g_e'(0,\eta)]\,d\eta = 0. \tag{6.13}$$

Having eqn (6.12), we now impose the requirement that both interior and edge-zone solution contributions participate in the satisfaction of this non-homogeneous edge condition. This means that we must have

$$V_i\rho^{1/2} = V_e. \tag{6.14}$$

Finally, we introduce a weighted average, β, of the variation of edge rotations in thickness direction by writing

$$\bar{u}_{,r} = -\beta s(\eta), \tag{6.15}$$

with

$$\int_{-1}^{1}(1-\eta^2)s(\eta)\,d\eta = \frac{4}{3}, \qquad \beta = -\frac{3}{4}\int_{-1}^{1}(1-\eta^2)\bar{u}_{,r}\,d\eta, \tag{6.16}$$

(so that $\beta = \beta_o$ when $\bar{u}_{,r} = -\beta_o = \text{const}$).

With (6.14) and (6.15) and with the further stipulation that

$$V_e = c\beta, \tag{6.17}$$

the nonhomogeneous edge condition appears now in the form

$$F_i'(0) + \frac{\alpha^2\rho}{1+\rho\eta} g_i'(0,\eta) + F_e'(0) + \frac{\alpha^2}{1+\rho\eta} g_e'(0,\eta) = s(\eta) \tag{6.18}$$

and the homogeneous edge conditions (6.13) and (6.10a) become

$$F_i(0) + \rho^{1/2}F_e(0) = 0, \tag{6.19a}$$

or

$$\int_{-1}^{1} [\rho g_i'(0,\eta) + g_e'(0,\eta)]\,d\eta = 0. \tag{6.19b}$$

Equations (6.18) and (6.19) are to be used in conjunction with the consequences of introducing (6.11) into eqns (6.3)–(6.6), that is, in conjunction with two systems which may be written in the form

$$\begin{aligned}&\int_{-1}^{1} g_i''\,d\eta + \frac{1-\nu^2}{\rho}\ln\frac{1+\rho}{1-\rho}F_i + \rho\int_{-1}^{1}\frac{\nu g_i\,d\eta}{(1+\rho\eta)^2} = 0\\ &g_i^{\cdot\cdot} + F_i'' + \rho\left[\alpha^2 g_i'' + \eta F_i'' - \frac{g_i^{\cdot} - \nu F_i}{1+\rho\eta}\right] = 0,\end{aligned} \tag{6.20}$$

and

$$\begin{aligned}&\int_{-1}^{1} g_e''\,d\eta + \rho^2\left[\int_{-1}^{1}\frac{\nu g_e\,d\eta}{(1+\rho\eta)^2} + \frac{1-\nu^2}{\rho}\ln\frac{1+\rho}{1-\rho}F_e\right] = 0\\ &g_e^{\cdot\cdot} + \alpha^2 g_e'' + F_e'' + \rho\left[-\frac{g_e^{\cdot}}{1+\rho\eta} + \eta F_e'' + \frac{\rho\nu F_e}{1+\rho\eta}\right] = 0.\end{aligned} \tag{6.21}$$

The solution of (6.20) and (6.21), subject to the edge conditions (6.18), (6.19) and (6.8) and subject to the face boundary conditions (6.7) is to be introduced into eqns (6.2) for Ψ and V. Upon taking account of the relations (6.11), (6.14) and (6.17) we have then

$$(1-\nu^2)\Psi = Ec\beta(g_e + \rho^{1/2}g_i), \qquad V = (ca)^{1/2}\beta(F_i + \rho^{1/2}F_e). \tag{6.22}$$

Furthermore, introduction of Ψ and V from eqn (6.2) into eqns (5.10)–(5.12) gives as expressions for stresses

$$\sigma_x = \frac{E\beta}{1-\nu^2}\frac{g_e^{\cdot} + \rho^{1/2}g_i^{\cdot}}{1+\rho\eta}, \quad \tau = -\frac{E\beta}{1-\nu^2}\frac{g_e' + \rho g_i'}{1+\rho\eta}, \tag{6.23}$$

$$\sigma_\theta = E\beta\left[\frac{\nu}{1-\nu^2}\frac{g_e^{\cdot} + \rho^{1/2}g_i^{\cdot}}{1+\rho\eta} + \frac{\rho^{1/2}F_i + \rho F_e}{1+\rho\eta}\right], \tag{6.24}$$

and as expressions for the stress couple M_x and the stress resultant Q,

$$M_x = -\frac{E\beta c^2}{1-\nu^2}\int_{-1}^{1}(g_e + \rho^{1/2}g_i)\,d\eta, \tag{6.25}$$

$$Q = -\frac{E\beta c}{1-\nu^2}\int_{-1}^{1}(g_e' + \rho g_i')\,d\eta. \tag{6.26}$$

We note that while the above formulas for M_x and Q give the impression that the edge-zone solution contribution g_e makes a dominant contribution to these quantities—which, if true would detract much from the significance of conventional two-dimensional shell theory—we will find shortly that $\int_{-1}^{1} g_e \, d\eta$ comes out to be of a smaller order of magnitude in ρ than $\rho^{1/2} \int_{-1}^{1} g_i \, d\eta$, so that no problem of this sort does, in fact, arise.

Beyond this we may also note that the differential equation for g_e in (6.21) is effectively of second order, in place of a fourth-order bi-harmonic problem which is encountered in the analysis of the corresponding problem for an isotropic homogeneous shell medium. It is this difference in the order of the differential-equation problem for the edge-zone solution contribution which makes possible an exact solution of the complete problem for the case of the transversely inextensional medium.

PARAMETRIC EXPANSIONS FOR INTERIOR AND EDGE-ZONE SOLUTION CONTRIBUTIONS

An inspection of the differential equations (6.20) and (6.21), in conjunction with the boundary conditions (6.7), (6.8) (6.18) and (6.19) indicates the possibility of an expansion of the solutions in powers of the parameter $\rho^{1/2}$†. Writing

$$F = F_0 + \rho^{1/2} F_1 + \rho F_2 + \dots, \qquad g = g_0 + \rho^{1/2} g_1 + \rho g_2 + \dots, \tag{7.1}$$

and observing that $(1/\rho) \ln [(1+\rho)/(1-\rho)] = 2[1 + \rho^2/3 + \dots]$, we have then from eqns (6.20),

$$\int_{-1}^{1} g''_{in} \, d\eta + 2(1-\nu^2) F_{in} = 0, \qquad \ddot{g}_{in} + F''_{in} = 0; \qquad n = 0, 1, \tag{7.2}$$

$$\left.\begin{aligned} &\int_{-1}^{1} g''_{in} \, d\eta + 2(1-\nu^2) F_{in} = -\nu \int_{-1}^{1} g_{in-2} \, d\eta; \\ &\ddot{g}_{in} + F''_{in} = \dot{g}_{in-2} - d^2 \dot{g}_{in-2} - \eta \dot{F}_{in-2} - \nu F_{in-2}; \end{aligned} \quad n = 2, 3 \right\}, \tag{7.3}$$

with analogous relations for $n = 4, 5, \dots$ which will not be utilized in what follows, and eqns (6.21) give in the same manner

$$\int_{-1}^{1} g''_{en} \, d\eta = 0, \quad \ddot{g}_{en} + \alpha^2 g''_{en} + \dot{F}_{en} = 0; \quad n = 0, 1, \tag{7.4}$$

$$\int_{-1}^{1} g''_{en} \, d\eta = 0, \quad \ddot{g}_{en} + \alpha^2 g''_{en} + \dot{F}_{en} = \dot{g}_{en-2} - \eta \dot{F}_{en-2}; \quad n = 2, 3, \tag{7.5}$$

$$\int_{-1}^{1} g''_{en} \, d\eta = -\nu \int_{-1}^{1} g_{en-4} \, d\eta - 2(1-\nu^2) F_{en-4}; \quad n = 4, 5. \tag{7.6‡}$$

The associated boundary conditions which follow from (6.7) and (6.8) are

$$\eta = \pm 1: \qquad g_{in} = 0, \qquad g_{en} = 0, \tag{7.7}$$

$$\xi = \infty: \qquad g_{in} = 0, \qquad g_{en} = 0, \qquad F_{in} = 0, \qquad F_{en} = 0. \tag{7.8}$$

The coupling between interior and edge zone contributions will be effected in the process of satisfying the boundary conditions for the loaded edge of the shell which follow from (6.18) and (6.19) in the form

$$F'_{i0}(0) + F'_{e0}(0) + \alpha^2 g'_{e0}(0, \eta) = s(\eta), \tag{7.9a}$$

$$F'_{i1}(0) + F'_{e1}(0) + \alpha^2 g'_{e1}(0, \eta) = 0, \tag{7.9b}$$

$$F'_{in}(0) + F'_{en}(0) + \alpha^2 g'_{en}(0, \eta) = \alpha^2 \eta g'_{en-2}(0, \eta) - \alpha^2 g'_{in-2}(0, \eta); \qquad n = 2, 3, \tag{7.9c}$$

†Except for the form of (6.19a) this expansion would be in powers of ρ, rather than $\rho^{1/2}$. This means that when (6.19b) applies, one-half the terms in our expansion will vanish automatically.

‡It is in this one place that we require equations for $n = 4, 5$, in order to complete the determination of our expansions terms up to $n = 3$.

and

$$F_{io}(0) = 0, \quad F_{in}(0) + F_{en-1}(0) = 0; \quad n = 1, 2, 3, \ldots \tag{7.10}$$

or

$$\int_{-1}^{1} g'_{en}(0, \eta)\, d\eta = 0; \qquad n = 0, 1, \tag{7.11a}$$

$$\int_{-1}^{1} g'_{en}(0, \eta)\, d\eta = -\int_{-1}^{1} g'_{in-2}(0, \eta)\, d\eta; \qquad n = 2, 3, \ldots . \tag{7.11b}$$

DETERMINATION OF INTERIOR SOLUTION CONTRIBUTION TERMS

We begin by determining the interior solution contribution to the extent that this is possible without bringing in the form of the associated edge zone solution contribution. In doing this we will arrive at a set of *ordinary* differential equations for the dimensionless deflection functions F_{in}, with F_{io} being equivalent to the corresponding function in accordance with conventional two-dimensional shell theory, with F_{i1} accounting for the effect of deviating from the "conventional" assumption of an η-independent edge slope function $s(\eta)$, and with F_{i2} accounting for the effects of transverse shear deformation and of element width changes in the same manner as previously found in [1].

We complement these results by the discovery of the *complete* system of boundary conditions for F_{io}, *for an arbitrarily prescribed edge slope function* $s(\eta)$, through use of the differential equations for the edge-zone solution contributions, but *without having to determine any part of the edge-zone portion* of the solution of the complete problem.

Considering the fact that $F_{in} = F_{in}(\xi)$, we find from eqns (7.2), in conjunction with the appropriate statement in (7.7) the simple result

$$g_{in} = \frac{1-\eta^2}{2} F''_{in}, \quad F^{IV}_{in} + 3(1-\nu^2)F_{in} = 0; \quad n = 0, 1. \tag{8.1}$$

In the same way we find from eqns (7.3) and (7.7), and with $(1-\nu^2)\alpha^2 = E/G$,

$$\left.\begin{aligned} g_{in} &= \frac{1-\eta^2}{2} F''_{in} + \frac{\eta-\eta^3}{3} F'_{in-2} + \left(\nu\frac{1-\eta^2}{2} - \frac{E}{G}\frac{5-6\eta^2+\eta^4}{8}\right) F_{in-2}, \\ F^{IV}_{in} &+ 3(1-\nu^2)F_{in} = \left(\frac{6E}{5G} - 2\nu\right) F''_{in-2}; \quad n = 2, 3. \end{aligned}\right\} \tag{8.2}$$

We now turn to the form of the boundary conditions for the functions F_{in}. We find from (7.8) the obvious requirements of vanishing F_{in} (and F'_{in}) for $\xi = \infty$. As far as conditions for $\xi = 0$ are concerned, we have as *one* of the two conditions for F_{io}, for one of the two cases considered in (7.10) and (7.11), that $F_{io}(0) = 0$. The other conditions, in accordance with (7.9)–(7.11) all appear to require the simultaneous determination of F_{en} and g_{en}. We will show next how to avoid this complication to a significant extent.

DERIVATION OF BOUNDARY CONDITIONS FOR F_{in} WITH OR WITHOUT INVOLVEMENT OF EDGE-ZONE SOLUTION CONTRIBUTION

We begin by observing that eqns (7.4) and (7.5), in conjunction with the conditions $g_{en}(\infty, \eta) = g'_{en}(\infty, \eta) = 0$ imply the relations

$$\int_{-1}^{1} g_{en}(\xi, \eta)\, d\eta = \int_{-1}^{1} g'_{en}(\xi, \eta)\, d\eta = 0, \tag{9.1a}$$

for $n = 0, 1, 2, 3$. Furthermore, from (6.21)

$$\int_{-1}^{1} g'_{e4}(\xi, \eta)\, d\eta = (1-\nu^2)\int_{\xi}^{\infty} F_{eo}(\xi)\, d\xi. \tag{9.1b}$$

With the help of (9.1), and through use of the first relation in (8.1) and (8.2), we may write the set of conditions in (7.10) and (7.11) in the form

$$F_{io}(0) = 0, \qquad or \;\; F'''_{io}(0) = 0, \tag{9.2a}$$

$$F_{i1}(0) = -F_{eo}(0), \qquad or \;\; F'''_{i1}(0) = 0, \tag{9.2b}$$

$$F_{in}(0) = -F_{en-1}(0), \quad or \;\; F'''_{in}(0) = \left(\frac{6E}{5G} - \nu\right) F'_{in}(0) - 3(1-\nu^2)\int_o^\infty F_{en-2}(\xi)\, d\xi; \qquad n = 2, 3. \tag{9.2c}$$

It remains to establish a *second* condition at $\xi = 0$ for F_{in}, through use of eqns (7.9), with a view towards letting this condition to be as much as possible a condition for $F'_{in}(0)$, in terms of given quantities. In order to accomplish this purpose, we consider an integrated version of the edge-zone differential equations (7.4) and (7.5), with an appropriate weighting function, as follows.

We write, as a consequence of eqn (7.4),

$$\int_{-1}^{1} (1-\eta^2)(\ddot{g}_{en} + \alpha^2 g''_{en} + F''_{en})\, d\eta = 0; \quad n = 0, 1. \tag{9.4}$$

In this we write now,

$$\int_{-1}^{1} (1-\eta^2)\ddot{g}_{en}\, d\eta = [(1-\eta^2)\dot{g}_{en} + 2\eta g_{en}]_{-1}^{1} - 2\int_{-1}^{1} g_{en}\, d\eta,$$

with the conclusion, which follows from (9.1) and (7.8), that

$$\int_{-1}^{1} (1-\eta^2)\, \ddot{g}_{en}\, d\eta = 0. \tag{9.5}$$

Introduction of (9.5) into (9.4) and observation of the edge conditions at infinity then gives further

$$\int_{-1}^{1} (1-\eta^2)(\alpha^2 g'_{en} + F'_{en})\, d\eta = 0; \quad n = 0,1. \tag{9.6}$$

Upon using (9.6), in conjunction with (7.9a, b), we obtain as conditions for F'_{io} and F'_{i1},

$$F'_{io}(0) = 1, \qquad F'_{i1}(0) = 0, \tag{9.7a,b}$$

and we see that, in fact, the determination of F_{io} may in *all* cases considered here be carried out without explicit reference to the edge zone contribution to the complete solution of the problem.

It suggests itself to see to what extent application of the same procedure to the differential equations (7.5) may make possible a reduction of the boundary conditions (7.9c) which involve F'_{i2} and F'_{i3}. We find first, proceeding in the same way as in going from (9.4) to (9.6), and observing that $\eta F''_{en-2}$ is an odd function of η, that now

$$\int_{-1}^{1} (1-\eta^2)(\alpha^2 g''_{en} + F''_{en})\, d\eta = 2\int_{-1}^{1} \eta g_{en-2}\, d\eta, \tag{9.8}$$

for $n = 2, 3$.

Evidently, eqn (9.8), together with the edge conditions at infinity, allows us to conclude that

$$\int_{-1}^{1} (1-\eta^2)\,[\alpha^2 g'_{en}(0,\eta) + F'_{en}(0)]\, d\eta = -2\int_{-1}^{1} \eta \int_0^\infty g_{en-2}(\xi,\eta)\, d\xi d\eta. \tag{9.9}$$

for $n = 2, 3$. Therewith, and with the consequences of the first relation in (8.1), we obtain altogether that

$$F'_{in}(0) = -\frac{2\alpha^2}{5} F'''_{in-2}(0) + \frac{3}{2}\int_{-1}^{1} \eta \int_{o}^{\infty} g_{en-2}(\xi, \eta)\, d\xi\, d\eta + \frac{3\alpha^2}{4}\int_{-1}^{1} (\eta - \eta^3)\, g'_{en-3}(0, \eta)\, d\eta, \tag{9.10}$$

for $n = 2, 3$. We note as a very important consequence of (9.10) the fact that *when g_{eo} is an even function of η*, which is the case when $s(\eta)$ is even, *then the integral terms on the right vanish for $n = 2$ and we have a boundary condition for F_{i2} as well which is free of any reference to the edge-zone solution contribution.*

EXPANSIONS FOR DEFLECTION, AXIAL BENDING MOMENT AND TRANSVERSE SHEAR STRESS RESULTANT

In order to see the possibility of determining M_x and Q, and also V, for the conventional displacement boundary condition case of an η-independent $\bar{u}_{,r}$ upto terms of relative order ρ, solely on the basis of a determination of the interior solution contribution, we rewrite our earlier expressions for these quantities as follows.

On the basis of eqns (6.22) and (7.1),

$$V = -\beta c^{1/2} a^{1/2}[F_{io} + \rho^{1/2}(F_{i1} + F_{eo}) + \rho(F_{i2} + F_{e1}) + \dots]. \tag{10.1}$$

On the basis of (6.25), (6.26) and (7.1), in conjunction with eqns (8.1) and (8.2) for the g_{in} and eqns (9.1) for the g_{en}

$$M_x = -\frac{2}{3}\frac{E\beta c^{5/2}}{(1-\nu^2)a^{1/2}}\left\{F''_{io} + \rho^{1/2}F''_{i1} + \rho\left[F''_{i2} - \left(\frac{6E}{5G} - \nu\right)F_{io}\right] + \dots\right\}, \tag{10.2}$$

$$Q = -\frac{2}{3}\frac{E\beta c^2}{(1-\nu^2)a}\left\{F'''_{io} + \rho^{1/2}F'''_{i1} + \rho\left[F'''_{i2} - \left(\frac{6E}{5G} - \nu\right)F'_{io} + 3\frac{1-\nu^2}{2}\int_{\xi}^{\infty} F_{eo}\, d\xi\right] + \dots\right\}. \tag{10.3}$$

We note that it will be possible to determine V, M_x and Q, upto terms of relative order ρ, soley on the basis of a consideration of the interior solution contributions F_{io}, F_{i1} and F_{i2}, to the extent that it is possible to determine these functions without explicit consideration of the edge-zone solution portion, as discussed in the preceding section, and to the extent that F_{eo} and F_{e1} turn out to vanish altogether.

The same conclusions apply insofar as the determination of the stiffness coefficients $K_{\beta\beta}$, $K_{V\beta}$ and K_{β} is concerned, inasmuch as we have that

$$K_{\beta\beta} = M_x(0)/\beta, \qquad K_{V\beta} = -Q(0)/\beta, \tag{10.4}$$

when $V(0) = 0$, and

$$K_{\beta} = M_x(0)/\beta, \tag{10.5}$$

when $Q(0) = 0$†.

EXPANSIONS FOR STRESSES

Introduction of the series expansions (7.1) into eqns (6.23) and (6.24) gives as expansions for the stresses σ_x, τ, σ_θ, up to terms of relative order ρ

$$\sigma_x = \frac{E\beta}{1-\nu^2}[\dot{g}_{eo} + \rho^{1/2}(\dot{g}_{e1} + \dot{g}_{io}) + \rho(\dot{g}_{e2} + \dot{g}_{i1} - \eta\dot{g}_{eo}) + \dots], \tag{11.1}$$

†Note that (10.4) and (10.5) are consistent with (4.1) and (4.2) in the light of the defining relation (6.16) for β.

$$\tau = -\frac{E\beta}{1-\nu^2}[g'_{eo} + \rho^{1/2}g'_{e1} + \rho(g'_{e2} + g'_{io} - \eta g'_{eo}) + \ldots], \tag{11.2}$$

$$\sigma_\theta = \nu\sigma_x + E\beta\,[\rho^{1/2}F_{io} + \rho(F_{i1} + F_{eo}) + \ldots], \tag{11.3}$$

We note that, in contrast to the results for *deflection*, axial *moment* and transverse *resultant*, the distribution of *stress* in the shell will, for the *general* case, be such that edge zone contributions come out to be of a higher order of magnitude than interior region contributions. This is not so for the "exceptional" cases for which g_{eo} and g_{e1} vanish identically. The significance of this distinction will become apparent in our discussion of the exceptional case of an η-independent $\bar{u}_{,r}$, in comparison with the case where $\bar{u}_{,r}$ is assumed to be a quadratic function of η.

SOLUTION OF INTERIOR AND EDGE-ZONE DIFFERENTIAL EQUATIONS

Determination of the interior solution contribution, in accordance with eqns (8.1) and (8.2), is the same as in the earlier work concerned with this contribution only[1]. We find, upon taking account once for all of conditions for $\xi = \infty$, with $4m^4 = 3(1-\nu^2)$, that

$$F_{in} = (A_n \cos m\xi + B_n \sin m\xi)\, e^{-m\xi}; \qquad n = 0, 1, \tag{12.1}$$

and

$$F_{in} = (A_n \cos m\xi + B_n \sin m\xi)\, e^{-m\xi} - \left(\frac{6E}{5G} - 2\nu\right)\frac{\xi}{16m^4}F'''_{in-2}; \qquad n = 2, 3, \tag{12.2}$$

with the associated functions g_{in} following from the above in accordance with (8.1) and (8.2).

Having F_{in} and g_{in}, as indicated above we now turn to the determination of F_{en} and g_{en}, for $n = 0, 1, 2, 3$, in accordance with eqns (7.4)–(7.8). Writing as before as differential equations for g_{en} and F_{en}, the set

$$\ddot{g}_{en} + \alpha^2 g''_{en} + F''_{en} = 0; \qquad n = 0, 1, \tag{12.3}$$

$$\ddot{g}_{en} + \alpha^2 g''_{en} + F''_{en} = \dot{g}_{en-2} - \eta F''_{en-2}; \qquad n = 2, 3, \tag{12.4}$$

we take account of the first relations in (7.4) and (7.5), in conjunction with the conditions for $\xi = \infty$, so as to establish as a system of *three* boundary conditions for $\eta = \pm 1$,

$$\int_{-1}^{1} g_{en}(\xi, \eta)\, d\eta = 0, \qquad g_{en}(\xi, \pm 1) = 0, \tag{12.5}$$

with the presence of the terms $F''_{en}(\xi)$ in (12.3) and (12.4) making it possible to satisfy the three conditions in (12.5), when without these terms the first condition (12.5) would rule out significant portions of the solutions of (12.3) and (12.4).

To see that this is in fact so we write (12.3) and (12.4) in the differentiated form

$$\dddot{g}_{en} + \alpha^2 \dot{g}''_{en} = 0; \qquad n = 0, 1, \tag{12.6}$$

$$\dddot{g}_{en} + \alpha^2 \dot{g}''_{en} = \ddot{g}_{en-2} - F''_{en-2}; \qquad n = 2, 3, \tag{12.7}$$

and determine first the solution of (12.6).

We find, by separation of variables, that eqn (12.6) has particular solutions

$$g_{enp} = (C_1 + C_2 \cos \lambda\eta)\, e^{-\lambda\xi/\alpha} + (D \sin \mu\eta)\, e^{-\mu\xi/\alpha}, \tag{12.8}$$

where C_1, C_2, D, λ and μ are arbitrary constants. Satisfaction of the three boundary conditions (12.5) requires that these arbitrary constants satisfy the relations

$$C_1 + C_2\lambda^{-1}\sin\lambda = 0, \qquad C_1 + C_2\cos\lambda = 0, \qquad D\sin\mu = 0. \tag{12.9}$$

The conditions for non-trivial solutions in (12.9) are

$$\tan\lambda = \lambda, \quad \sin\mu = 0, \tag{12.10}$$

for all positive roots $\lambda = \lambda_j$ and $\mu = \mu_k$ for $j,k = 1,2,3,\ldots$. It is evident that the functions sin $\mu_k\eta$ form an *orthonormal* system, such that every integrable *odd* function of η in the interval $-1 < \eta < 1$ may be expanded in terms of them. Additionally, it can be shown that the set

$$\phi_j = \frac{\cos\lambda_j - \cos\lambda_j\eta}{\lambda_j\cos\lambda_j} \tag{12.11}$$

also forms an orthonormal system, in such a way that every integrable *even* function $f(\eta)$, in $-1 < \eta < 1$, with the additional property $\int_{-1}^{1} f\,d\eta = 0$, can be expanded in terms of them†.

We use the above properties to construct the series solution

$$g_{en} = \sum_{j,k=1}^{\infty} C_{jn}\varphi_j(\eta)\,e^{-\lambda_j\beta/\alpha} + D_{kn}(\sin\mu_k\eta)\,e^{-\mu_k\beta/\alpha} \tag{12.12}$$

for $n = 0, 1$.

Before proceeding to the analogous solution of (12.7) we return to eqn (12.3) in order to establish, with the help of the conditions $F_{en}(\infty) = F'_{en}(\infty) = 0$, as solution $F_{en}(\xi)$ which is associated with g_{en} in (12.12),

$$F_{en} = -\alpha^2\sum_{j=1}^{\infty}\lambda_j^{-1}C_{jn}\,e^{-\lambda_j\beta/\alpha}; \qquad n = 0, 1. \tag{12.13}$$

We now consider eqn (12.4) where it remains to determine the appropriate particular solution corresponding to a r.h.s. which can be seen, with the help of (12.11)–(12.13), to be of the form

$$\sum C_{jn-2}\left(\frac{\sin\lambda_j\eta}{\cos\lambda_j} + \lambda_j\eta\right)e^{-\lambda_j\beta/\alpha} + D_{kn-2}(\mu_k\cos\mu_k\eta)\,e^{-\mu_k\beta/\alpha}.$$

We find that a particular solution of (12.4) with the above expression in place of the original r.h.s., which also satisfies the three boundary conditions (12.5), is such that altogether

$$g_{en} = \sum(C_{jn}\varphi_j(\eta)\,e^{-\lambda_j\beta/\alpha} + D_{kn}\sin\mu_k\eta\,e^{-\mu_k\beta/\alpha}) + \sum C_{jn-2}\left(\frac{\eta}{\lambda_j} - \frac{\eta\cos\lambda_j\eta}{2\lambda_j\cos\lambda_j} - \frac{\sin\lambda_j\eta}{2\lambda_j\sin\lambda_j}\right)e^{-\mu_k\beta/\alpha}$$
$$+\frac{1}{2}\sum D_{kn-2}\left(\eta\sin\mu_k\eta - \frac{\cos\mu_k\eta - \cos\mu_k}{\mu_k}\right)e^{-\mu_k\beta/\alpha}, \tag{12.14}$$

and

$$F_{en} = -\alpha^2\sum\lambda_j^{-1}C_{jn}\,e^{-\lambda_j\beta/\alpha} - \frac{1}{2}\alpha^2\sum\mu_k^{-1}\cos\mu_k D_{kn-2}\,e^{-\mu_k\beta/\alpha}, \quad \text{for } n = 2, 3. \tag{12.15}$$

It now remains to determine the constants of integration C_{jn} and D_{kn} in conjunction with, or preferrably subsequent to, the determination of the constants A_n and B_n in the interior solution portion F_{in} as given by (12.1) and (12.2).

DERIVATION OF BOUNDARY CONDITIONS FOR DETERMINATION OF CONSTANTS IN g_{en}

We begin by recalling that the complete system (7.9)–(7.11) of conditions at the edge $\xi_e = \xi_i = 0$ of the shell had been reduced, insofar as possible, to a system of conditions for the

†We note that it is this orthogonality property of the functions φ_j which is responsible for the advantages of the present solution of the general problem, in comparison with the solution of the corresponding problem for the isotropic shell.

F_{in} alone, consisting of eqns (9.2), (9.3), (9.7) and (9.10). A characteristic of these conditions is that in them the effect of the edge-zone solution contribution manifests itself through the presence of certain definite integrals with respect to η of $g_{en}(\xi, \eta)$, but without appearance of the g_{en} themselves. Consequently, these conditions do not contain those elements of the complete system of edge conditions (7.9)–(7.11) which are required for a determination of the series coefficients C_{jn} and D_{kn} in g_{en}.

A re-inspection of eqns (7.9)–(7.11) makes it clear that the remaining supplementation of the boundary conditions (9.2), (9.3), (9.7) and (9.10) must come from eqns (7.9), and not from (7.10) and (7.11).

We obtain the first of the remaining transformed conditions by introducing (9.7a) into (7.9a) and by eliminating $F'_{io}(0)$ from this relation by considering that $\int_{-1}^{1} g'_{eo}\,d\eta = 0$. This gives

$$\alpha^2 g'_{eo}(0, \eta) = s(\eta) - \frac{1}{2}\int_{-1}^{1} s(\eta)\,d\eta. \tag{13.1}$$

We finally introduce $F'_{in}(0)$ from (9.10) into (7.9c) and consider that here $\int_{-1}^{1} g'_{en}\,d\eta = 0$, and $g'_{in-2}(0, \eta) = (1/2)(1-\eta)\,F'''_{in-2}(0)$. Therewith eqn (7.9c) becomes

$$g'_{en}(0, \eta) = \eta g'_{en-2}(0, \eta) - \frac{1}{2}\int_{-1}^{1} \eta g'_{en-2}(0, \eta)\,d\eta - \frac{1}{2}\left(\frac{1}{3} - \eta^2\right) F'''_{in-2}(0), \quad \text{for } n = 2, 3. \tag{13.3}$$

Having eqns (13.1)–(13.3), in conjunction with eqns (12.12) and (12.13) for g_{en} and F_{en}, we are now in a position to arrive at some general conclusions, without additional analysis. To wit

(1) The functions g_{e1} and F_{e1} will vanish identically, for all edge conditions cases here under consideration.

(2) The functions g_{eo} and F_{eo} will vanish identically for the special case $s(\eta) = 1$, but will otherwise have a dominant effect in eqns (11.1)–(11.3) for the distribution of stress, and a direct effect of relative order $\rho^{1/2}$ on the values of the deflection V, in accordance with eqn (10.1), as well as an indirect effect of relative order $\rho^{1/2}$ on M_x and Q, in accordance with eqns (10.2), (10.3) and (9.2b).

(3) The functions g_{e2} and F_{e2} will make a contribution of relative order ρ on stresses, in accordance with (11.1)–(11.3) but will make no such contribution to V, M_x and Q.

THE CASE OF A UNIFORM EDGE ROTATION

We assume now, as in [4], an axial-edge displacement $\bar{u} = -\beta_o(r-a)$, giving $\bar{u}_{,r} = -\beta_o$ and, in accordance with (6.16), $\beta = -(3/4)(-\beta_o)(4/3) = \beta_o$ and therewith, in accordance with (6.15),

$$s = 1, \tag{14.1}$$

and then, as discussed in the preceding section

$$g_{eo} = 0, \qquad F_{eo} = 0. \tag{14.2}$$

With (14.2), and with $g_{e1} = 0$ and $F_{e1} = 0$, we have, from (9.2), (9.3), (9.7) and (9.10) as edge conditions for the interior solution contribution

$$F'_{io}(0) = 1, \quad F'_{i1}(0) = F'_{i2}(0) = F'_{i3}(0) = 0 \tag{14.3}$$

together with

$$F_{io}(0) = F_{i1}(0) = F_{i2}(0) = F_{i3} = 0, \tag{14.4}$$

for the case $V(0) = 0$, and

$$F'''_{io}(0) = F'''_{i1}(0) = F'''_{i2}(0) = F'''_{i3}(0) = 0 \tag{14.5}$$

for the case $Q(0) = 0$, with eqn (14.3), together with (14.4) or (14.5), to be used for the determination of the constants of integration A_n and B_n in (12.1) and (12.2).

We omit the elementary calculations leading to the appropriate values of the A_n and B_n. With their help there follows

$$V(0) = -c^{1/2}a^{1/2}\beta\,[F_{io}(0) + \rho F_{i2}(0) + \ldots]$$
$$= \frac{c^{1/2}a^{1/2}}{2m}\beta\left[1 - \frac{\rho}{m^2}\left(\frac{3E}{20G} + \frac{\nu}{4}\right) + \ldots\right], \tag{14.6}$$

$$M_x(0) = -\frac{Ec^{5/2}}{2m^4a^{1/2}}\beta\left\{F''_{io}(0) + \rho\left[F''_{i2}(0) - \left(\frac{6E}{5G} - \nu\right)F_{io}(0)\right] + \ldots\right\}$$
$$= \frac{Ec^{5/2}}{2m^3a^{1/2}}\beta\left[1 - \frac{\rho}{m^2}\left(\frac{3E}{20G} - \frac{\nu}{4}\right) + \ldots\right], \tag{14.7}$$

for the case $Q(0) = 0$, and

$$M_x(0) = \frac{Ec^{5/2}}{m^3a^{1/2}}\beta\left[1 - \frac{\rho}{m^2}\left(\frac{9E}{20G} + \frac{\nu}{4}\right) + \ldots\right], \tag{14.8}$$

$$Q(0) = -\frac{Ec^2}{m^2a}\beta\left[1 - \frac{\rho}{m^2}\left(\frac{3E}{5G} + \frac{\nu}{2}\right) + \ldots\right], \tag{14.9}$$

for the case $V(0) = 0$ where, it is recalled, $4m^4 = 3(1-\nu^2)$ and $\rho = c/a$.

We note, specifically, that the results in (14.6)–(14.9) have been obtained without determination of any edge-zone solution contribution and that furthermore the stiffness coefficients $K_{\beta\beta}$ and $K_{V\beta}$ which are associated with (14.8), in accordance with (10.4), as well as the coefficient K_β associated with (14.7) in accordance with (10.5) are consistent with our earlier bound results in [4], upon specializing these so as to correspond to the limiting-type assumption of a medium unable to experience transverse normal strains.

Having determined the F_{in}, we may now obtain the here leading terms g_{e2} and F_{e2} of the edge-zone solution portion of the complete solution of the problem from eqns (12.14) and (12.15), with $C_{jo} = 0$ and $D_{jo} = 0$, in conjunction with the boundary condition (13.3) which now reduces to the form

$$g'_{e2}(0, \eta) = \frac{1}{2}\left(\frac{1}{3} - \eta^2\right)F'''_{io}(0), \tag{14.10}$$

or, equivalently, to

$$\sum \lambda_j C_{j2}\,\varphi_j + \mu_k D_{k2}\sin\mu_k\eta = \frac{1}{2}\alpha\left(\eta^2 - \frac{1}{3}\right)F'''_{io}(0) \tag{14.11}$$

in the interval $-1 < \eta < 1$. The orthonormality properties of the functions φ_j and $\sin\mu_k\eta$ then give

$$D_{k2} = 0, \quad \lambda_j C_{j2} = \frac{\alpha}{2}F'''_{io}(0)\int_{-1}^{1}\left(\eta^2 - \frac{1}{3}\right)\varphi_j\,d\eta = \frac{2\alpha}{3\lambda_j}F'''_{io}(0) \tag{14.12}$$

and therewith

$$g_{e2} = \frac{2}{3}\alpha\left(\sum \lambda_j^{-2}\varphi_j(\eta)\,e^{-\lambda_j\beta\alpha}\right)F'''_{io}(0). \tag{14.13}$$

A consideration of (14.5) now gives that g_{e2} vanishes throughout, just as g_{eo} and g_{e1}, for the case $Q(0) = 0$. For the case $V(0) = 0$ the function F_{io}, with $F'_{io}(0) = 1$ and $F_{io}(0) = 0$, gives the relation $F'''_{io}(0) = 2m^2$ for the last factor in the expression for g_{e2}.

Having determined g_{e2} it becomes possible to evaluate the distribution of stress in the shell, in accordance with eqns (11.1)–(11.3), with these equations reducing, for the present problem to the form

$$\sigma_x = \frac{E\beta\rho^{1/2}}{1-\nu^2}[\dot{g}_{io} + \rho^{1/2}\dot{g}_{e2} + \ldots], \quad \tau = -\frac{E\beta\rho}{1-\nu^2}[g'_{io} + g'_{e2} + \ldots] \tag{14.14}$$

with a corresponding expression for σ_θ. We see once again that the "elementary" interior contribution in σ_x (as well as in σ_θ) dominates the supplementary edge zone contribution. At the same time the edge zone contribution is of the same order of magnitude as the interior contribution insofar as the transverse shearing stress τ is concerned. We note, in particular, the possibility of writing, on the basis (14.14) and (14.10), as expression for the *edge* shear distribution for the case $V(0) = 0$

$$\tau(0, \eta) \approx \frac{E\beta\rho}{1-\nu^2}\frac{m^2}{2}\left[(1-\eta^2) - \left(\frac{1}{3} - \eta^2\right)\right] \tag{14.15}$$

with the second term inside the braces representing the edge zone effect, which has the expected property of making no contribution to the shear stress resultant Q.

A CASE OF NON-UNIFORM EDGE ROTATION

We now consider, as an example for which the results of the standard two-dimensional shell theory are complemented in an essential way by three-dimensional considerations, the case for which the axial edge displacement is prescribed in the form

$$\bar{u} = -\beta_o\left[(r-a) + \frac{k}{3}\frac{(r-a)^3}{c^2}\right], \qquad \bar{u}_{,r} = -\beta_o[1 + k\eta^2] \tag{15.1}$$

and therewith, in accordance with (6.16) and (6.15)

$$\beta = -\beta_o\left(1 + \frac{1}{5}k\right), \qquad s(\eta) = \frac{1+k\eta^2}{1+k/5}. \tag{15.2}$$

We can now, as before, determine the *first* term of the interior solution contribution, with the help of the edge conditions

$$F'_{io}(0) = 1, \quad F_{io}(0) = 0 \quad or \quad F'''_{io}(0) = 0 \tag{15.3}$$

which follow from (9.2a) and (9.7a), depending on whether $V(0) = 0$ *or* $Q(0) = 0$ is prescribed, and we note that this determination *does* incorporate information on the nature of the shape functions in (15.1), by way of the defining relation (15.2) for β.

Introduction of g_{eo} from eqn (12.12) into the relation

$$\alpha^2 g'_{eo}(0) = k\frac{\eta^2 - 1/3}{1+k/5}, \tag{15.4}$$

which follows from (13.1) in conjunction with (15.2) now gives

$$C_{jo} = \frac{4}{\alpha\lambda_j^2}\frac{k/3}{1+k/5}, \qquad D_{ko} = 0. \tag{15.5}$$

Having g_{eo}, and g_{io}, and recalling that $g_{e1} = 0$ for all cases, we now have as expressions for stresses, upto and including terms of relative order $\rho^{1/2}$,

$$\sigma_x = \frac{E\beta}{1-\nu^2}(\dot{g}_{eo} + \rho^{1/2}\dot{g}_{io}), \quad \tau = \frac{-E\beta}{1-\nu^2}g'_{eo}, \tag{15.6}$$

as well as $\sigma_\theta = \nu\sigma_x + E\beta\rho^{1/2}F_{io}$ where, notably, all components of stress are, within the narrow edge zone, of a higher order of magnitude than the expected stresses associated with the interior solution contributions g_{io} and F_{io}.

We omit listing explicit formulas for stresses, and instead consider eqns (10.1)–(10.3) for V, M_x and Q, where we note that evaluation of the contributions of relative order $\rho^{1/2}$ and ρ involves the relations

$$F_{eo}(0) = -\frac{4\alpha k/3}{1+k/5}\sum\frac{1}{\lambda_j^3},\quad \int_0^\infty F_{eo}\,d\xi = -\frac{4\alpha^2 k/3}{1+k/5}\sum\frac{1}{\lambda_j^4}. \tag{15.7}$$

Upon evaluation of (10.1)–(10.3), we now obtain in generalization of eqns (14.6)–(14.9)

$$\frac{V(0)}{\sqrt{(ca)}} = \frac{\beta}{2m}\left\{1+\frac{\rho^{1/2}}{m}\sqrt{\left(\frac{E}{3G}\right)}\frac{4k}{1+k/5}\sum\frac{1}{\lambda_j^3}-\frac{\rho}{m^2}\left[\frac{3E}{20G}\left(1+\frac{40k/3}{1+k/5}\sum\frac{1}{\lambda_j^4}\right)+\frac{\nu}{4}\right]+\ldots\right\} \tag{15.8}$$

$$\frac{M_x(0)}{Ec^2} = \frac{\beta}{2m^3}\sqrt{\left(\frac{c}{a}\right)}\left\{1-\frac{\rho}{m^2}\left[\frac{3E}{20G}\left(1-\frac{40k/3}{1+k/5}\sum\frac{1}{\lambda_j^4}\right)-\frac{\nu}{4}\right]+\ldots\right\} \tag{15.9}$$

when $Q(0)=0$, and

$$\frac{M_x(0)}{Ec^2} = \frac{\beta}{m^3}\sqrt{\left(\frac{c}{a}\right)}\left\{1+\frac{\rho^{1/2}}{m}\sqrt{\left(\frac{E}{3G}\right)}\frac{2k}{1+k/5}\sum\frac{1}{\lambda_j^3}-\frac{\rho}{m^2}\left(\frac{9E}{20G}+\frac{\nu}{4}\right)+\ldots\right\} \tag{15.10}$$

$$\frac{Q(0)}{Ec} = -\frac{\beta}{m^2}\frac{c}{a}\left\{1+\frac{\rho^{1/2}}{m}\sqrt{\left(\frac{E}{3G}\right)}\frac{4k}{1+k/5}\sum\frac{1}{\lambda_j^3}-\frac{\rho}{m^2}\left[\frac{3E}{5G}\left(1+\frac{5k/3}{1+k/5}\sum\frac{1}{\lambda_j^4}\right)+\frac{\nu}{2}\right]+\ldots\right\} \tag{15.11}$$

when $V(0)=0$.

In order to evaluate (15.8)–(15.11), we note that with the successive roots $\lambda_j \approx 4.49$, 7.72, 10.90, 14.07, 17.22, 20.37, 23.52, 26.67, 29.81, 32.99, $(j+(1/2))\,\pi,\ldots$ of eqn (12.10) the two sums of negative powers of λ_j come out to be

$$\sum\lambda_j^{-3}\approx 0.015,\qquad \sum\lambda_j^{-4}\approx 0.0029. \tag{15.12}$$

Insofar as the interpretation of eqns (15.8)–(15.11) is concerned, it should be noted that the *leading terms* on the right may be considered as equivalent to the consequences of ordinary thin shell theory, *in conjunction with the solution of the problem of how to introduce an appropriate representation of the displacement condition* (15.1) *into this theory*. Additionally, we find that while the effect of transverse shear deformability and of cross-sectional width changes comes out, as expected, to be of relative order ρ, the effect of the η^2-term in $\bar{u}_{,r}$ comes out to be of order $\rho^{1/2}$, with the numerically largest values of these correction terms resulting upon letting k tend to infinity, with a finite limiting value of $\beta_0 k$.

To obtain an impression of the numerical consequences of replacing a linear distribution $\bar{u} = u_0(r-a)/c$ by a pure cubic distribution $\bar{u} = u_0(r-a)^3/c^3$ we set in eqns (15.8)–(15.11) $k=\infty$ and $\beta = -\beta_0 k/5 = 3u_0/5c$. Therewith, and with eqns (15.12), there follows

$$\frac{V(0)}{\sqrt{(ca)}} \approx \frac{\beta}{2m}\left\{1+0.3\frac{\rho^{1/2}}{m}\sqrt{\left(\frac{E}{3G}\right)}-\frac{\rho}{m^2}\left[\frac{3E}{20G}(1+0.2)+\frac{\nu}{4}\right]\right\} \tag{15.13}$$

$$\frac{M_x(0)}{Ec^2} \approx \frac{\beta}{2m^3}\sqrt{\left(\frac{c}{a}\right)}\left\{1-\frac{\rho}{m^2}\left[\frac{3E}{20G}(1-0.2)-\frac{\nu}{4}\right]\right\} \tag{15.14}$$

when $Q(0)=0$, and

$$\frac{M_x(0)}{Ec^2} \approx \frac{\beta}{m^3}\sqrt{\left(\frac{c}{a}\right)}\left\{1+0.15\frac{\rho^{1/2}}{m}\sqrt{\left(\frac{E}{3G}\right)}-\frac{\rho}{m^2}\left(\frac{9E}{20G}+\frac{\nu}{4}\right)\right\} \tag{15.15}$$

$$\frac{Q(0)}{Ec} \approx -\frac{\beta}{m^2}\frac{c}{a}\left\{1 + 0.3\frac{\rho^{1/2}}{m}\sqrt{\left(\frac{E}{3G}\right)} - \frac{\rho}{m^2}\left[\frac{3E}{5G}(1 + 0.025) + \frac{\nu}{2}\right]\right\} \tag{15.16}$$

when $V(0) = 0$.

As might be expected, the shape correction terms with $\rho^{1/2}$ come out to be numerically quite significant for moderately thin shells, say for $\rho = 0.1$. Additionally, due to the change from a linear to a cubic edge displacement distribution a significant modification of the terms with ρ, via the additive terms ± 0.2 and 0.025, is seen to occur in some of the above expressions.

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